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Modal Analysis of Gyroscopic Flexible Spacecraft: A Continuum Approach

Hari B. Hablani*
Johnson Space Center, Houston, Texas

In the past, modal analyses of gyroscopic spacecraft have been performed *after* constructing a discrete model of the dynamics. This, however, obscures any momentum interaction between different parts of the spacecraft. This paper uses a continuum approach and gives a clear formulation of the complex gyroscopic vehicle modes. It is shown that unlike discrete modal analysis, special care is required while composing the phase space representation of the vehicle dynamics for a continuum modal analysis. The motion of the vehicle is shown to consist of two phases. In the first phase the vehicle executes a rigid body rotation with respect to an inertial frame. This rotation depends on the instantaneous angular momentum of the vehicle and determines the location of the associated Tisserand's frame. The second phase accounts for deformation measured from the Tisserand's frame and is expressed as a superposition of infinite gyroscopic modes. This phase contributes zero overall angular momentum. Throughout the analysis, a complex inner product in Hilbert space is found to be an essential tool for analysis.

I. Introduction

THE difficulty with any gyroscopic spacecraft is that so far the associated vehicle modes have not been characterized clearly. When confronting such a spacecraft, previous investigators have first constructed a discrete model of the dynamics and then performed a phase space modal analysis; see, for example, Likins,[1] Meriovitch,[2] Hablani and Shrivastava.[3] However, such fundamental questions as "How much is the angular momentum in a phase space gyroscopic mode?" were never raised. It is circuitous to answer this and the related questions by the available[1-3] discrete techniques. In contrast, the present paper offers a direct continuum approach to formulate gyroscopic vehicle modes. Various basic questions related to the dynamics of flexible gyroscopic spacecraft are raised and answered in the paper.

To project the ideas of the paper with clarity, we will shortly define three families of modes. In the following the word "constrained" is used when a constraint of some nature restricts the motion of the spacecraft; similarly, the word "unconstrained" implies that the vehicle is free. Throughout the paper, a "simple" spacecraft will mean a nongyroscopic, nonspinning spacecraft. The definitions are as follows:

Simple constrained modes[4]: Modes associated with the elastic vibrations of a simple spacecraft whose rigid portion is constrained to be immobile.

Simple unconstrained modes[4]: Modes associated with the unrestrained vibrations of entire spacecraft; in this case different parts of the vehicle interact freely.

The simple constrained modes are generally used to analyze the dynamics of flexible spacecraft (see Likins,[5] for example). Meanwhile, the use of the simple unconstrained modes leading to math models of higher fidelity is increasing, see Ref. 6, where these modes are shown to play a more important role in control studies than the simple constrained modes.

To define the second family of modes, we note that the planar dynamics of the spinning spacecraft do not lead to the gyroscopic forces; the following two types of modes are related to such a situation. Consult Ref. 7 for some additional comments.

Spinning constrained modes: A mode in which a spinning elastic surface is caused to vibrate without disturbing the uniform spin and nominal attitude of the spacecraft. This mode may also be called "a uniform spin mode."

Spinning unconstrained modes: Like simple unconstrained modes, this is a vehicle mode portraying the unrestricted planar dynamics of the vehicle; consequently, the spinning rigid portion and the deformation of the elastic members exchange energy and momentum freely.

An example of the use of the spinning constrained modes while determining the stability of a flexible spinning satellite can be seen in Hughes and Fung.[8] On the other hand, Austin and Pan[9] were first in composing the spinning unconstrained modes for a rotating beam with tip masses having rotary inertia.

The third family of the modes is concerned with the gyroscopic situations. The corresponding constrained modes might be defined such that they are any of the previously mentioned modes. However, the unconstrained modes entitled *gyroscopic vehicle modes* or *gyroscopic modes*, for brevity, are substantially more involved than the above four types of modes and are definable in a phase space only. We will now briefly sketch the present contribution.

This paper presents modal analyses of a model under four situations of increasing complexity, treating only the unconstrained modes. The model consists of a central rigid body $\mathcal{R}$ and an elastic beam $\mathcal{E}$; a study of the interaction between $\mathcal{R}$ and $\mathcal{E}$ is desired in all situations. In a differential formulation of the vibrations of such a system, boundary conditions involve eigenvalues, complicating the modal analysis. We illustrate that depending on the complexity of the eigenvalue problem one has to carefully devise a scalar product in a function space to obtain the orthogonal modes. Section II investigates the self-adjointness of the simple unconstrained modes. The spinning unconstrained modes are deduced in Sec. III for a gravity-stabilized satellite executing planar dynamics. In Sec. IV the dynamics of the above model, now spinning about the longitudinal axis and undergoing arbitrary deformations, is analyzed. In Sec. V we examine a situation in which the rigid body houses a rotor. In both Sec. IV and V a phase space modal analysis shows that the angular momentum associated with a gyroscopic mode is zero. From the examples we learn that the dynamics of the Tisserand's frame of any gyroscopic spacecraft in the presence of external stimuli is the same whether the vehicle is rigid or flexible. Section VI summarizes the results and their significance.

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*NRC/NASA Research Associate. AIAA Member.

II. Simple Unconstrained Modes

Here we are interested in a simple spacecraft because analysis of such spacecraft will help us devise the gyroscopic modes. The model described in Sec. I is shown in Fig. 1. The long, flexible free beam $\mathcal{E}(-\ell<x<-0,\ -0<x<\ell)$ has the rigid body $\mathcal{R}\ (-0<x<+0)$ at the center. A simple, but very interesting, example of the dynamics of a similar model may be reviewed in Sec. 7.17 of Ref. 10. Readers interested in the modal analysis of a two-dimensional platform-type structure may review Hablani.[11]

The influence of the appendage deformation on the rigid body can be modeled either by a displacement method or by a force method. Presently we will use the latter technique because the explicit consideration of the bending moments and the shear forces due to the deformations at the interface exposes the basic nature of the associated eigenvalue problems and proves to be pivotal in devising the gyroscopic modes. The linear equations of motion of the model shown in Fig. 1 are

$$x\epsilon\mathcal{R}:\ m_r\ddot{z}_c=EI(\Delta v'_- -\Delta v'_+),\quad A_r\ddot{\theta}=-EI(\Delta v_- -\Delta v_+) \tag{1a}$$

$$x\epsilon\mathcal{E}:\ k\Delta k\Delta v+(\ddot{z}_c+x\ddot{\theta}+\ddot{v})=0 \tag{1b}$$

$$x=-\ell:\ \Delta v=\Delta v'=0;\quad x=-0,+0:\ v+v'=0 \tag{1c}$$

where m_r and A_r are the mass of $\mathcal{R}$ and its moment of inertia about an axis normal to the plane of the deformation; $k^2=EI/\rho$; $\Delta=\partial^2/\partial x^2$. The subscripts "+" and "−", here and throughout, imply the value fo the function at the roots $+0,-0$, respectively. EI is the bending rigidity and ρ the mass/length of $\mathcal{E}$. (z_c,θ) are the translation and rotation of $\mathcal{R}$, and $v(x,t)$ is the transverse deformation of $\mathcal{E}$ with respect to a body-fixed frame. The primes and dots represent, as always, spatial and time derivatives. The displacement

$$w(x,t)\triangleq z_c(t)+x\theta(t)+v(x,t),x\epsilon\mathcal{V}(=\mathcal{R}+\mathcal{E}) \tag{2}$$

at any point x is with respect to an inertial frame passing through the mass center of $\mathcal{V}$ in its undisplaced, undeformed state.

Our aim is not a complete modal analysis of Eq. (1); for this the reader may refer to Hughes.[12] Instead, our interest lies in constructing a novel technique of the modal analysis of a free vehicle such as the one represented by Eq. (1). In planar dynamics, this vehicle has a translational and a rotational rigid mode which possess, respectively, linear and angular momentum of the vehicle. It is known that the free elastic modes which are othogonal to the rigid modes posses neither translational nor angular momentum.[12] Let $w_m(x)$ be one such (real) mode where, here and throughout, the subscript $m=1,\ldots,\infty$, unless stated otherwise. Equation (1b) can be assumed to have the following solution

$$w(x,t)=w_m(x)\eta_m(t) \tag{3}$$

By comparing Eq. (3) with Eq. (2), we get a similar solution for z_c,θ_x, and θ_y:

$$z_c(t)=z_{cm}\eta_m(t),\quad \theta(t)=\theta_m\eta_m(t),$$

$$v(x,t)=v_m(x)\eta_m(t), \tag{4a}$$

$$w_m(x)=z_{cm}+x\theta_m+v_m \tag{4b}$$

Equations (3) and (4) transform Eq. (1) by standard operations to the following eigenvalue problem

$$k\Delta k\Delta v_m=\omega_m^2 w_m \tag{5}$$

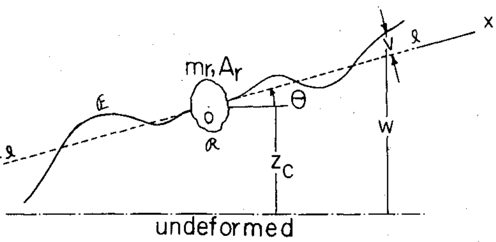

Fig. 1 Simple spacecraft in vibrations.

which has these boundary conditions

$$x=-\ell,\ell:\ \Delta v_m=\Delta v'_m=0;\qquad x=-0,+0:\ v_m=v'_m=0 \tag{6a}$$

$$-EI(\Delta v'_{m-}-\Delta v'_{m+})=\omega_m^2 m_r z_{cm},$$

$$EI(\Delta v_{m-}-\Delta v_{m+})=\omega_m^2 A_r\theta_m \tag{6b}$$

where $\Delta v_{m+}=v''_m(x=+0)$, etc. Note that the boundary conditions involve the eigenvalue ω_m^2.

To establish the orthogonality of the modes w_m and w_n, $m\neq n$, we examine the self-adjointness of the eigenvalue problem Eq. (5). Standard conditions exist for those eigenvalue problems in which the boundary conditions involve the eigenvalues, see Sec. 5.8 of Ref. 13 and Sec. 5.2 of Ref. 14. However, these conditions are restricted to the cases in which the rigid body and the elastic members have the same form of kinetic energy, whereas, presently, the rigid body has translational and rotational kinetic energy but the elastic structure has only the former. Consequently, we proceed as follows.

Define the following three-component vectors

$$\tilde{w}_m\triangleq\begin{bmatrix} w_m \\ \hline m_r^{1/2}z_{cm} \\ A_r^{1/2}\theta_m \end{bmatrix},$$

$$\tilde{L}w_m\triangleq\begin{bmatrix} k\Delta k\Delta v_m \\ \hline -EI(\Delta v'_{m-}-\Delta v'_{m+})m_r^{-1/2} \\ EI(\Delta v_{m-}-\Delta v_{m+})A_r^{-1/2} \end{bmatrix} \tag{7}$$

Note that second and higher spatial derivatives of v_m and w_m are the same. The partition in the above vectors separates field terms from boundary terms. The vectors in Eq. (7) are designed from the eigenvalue problem, Eq. (5), and the boundary conditions, Eq. (6), such that the following is valid

$$\tilde{L}w_m=\omega_m^2\tilde{w}_m \tag{8a}$$

$$\langle\tilde{w}_m,\tilde{L}w_n\rangle=\langle\tilde{L}w_m,\tilde{w}_n\rangle \tag{8b}$$

where

$$\langle w_m,\tilde{L}w_n\rangle\triangleq\int_{\mathcal{E}} w_m k\Delta k\Delta v_n \mathrm{d}m-z_{cm}EI(\Delta v'_{n-}-\Delta v'_{n+})$$

$$+\theta_m EI(\Delta v_{n-}-\Delta v_{n+}) \tag{8c}$$

The inner product $\langle\tilde{L}w_m,\tilde{w}_n\rangle$ can be defined similarly. Thus, Eq. (8b) proves the self-adjointness of the eigenvalue problem, Eqs. (5) and (6). This leads us, as the reader can show, to the following orthogonality condition

$$\langle\tilde{w}_m,\tilde{w}_n\rangle\triangleq\int_{\mathcal{E}} w_m w_n \mathrm{d}m+m_r z_{cm}z_{cn}+A_r\theta_m\theta_n=\delta_{mn} \tag{9}$$

where δ_{mn} is a Kronecker delta. This orthonormality condition is a variant of the general conditions, Eq. (62), given by

Hughes[12] for a *simple* spacecraft.

To solve Eq. (5) one can eliminate z_{cm} and θ_m with the help of Eq. (6) to obtain

$$k\Delta k\Delta v_m + EI[(\Delta v'_{m-} - \Delta v'_{m+})/m_r - x(\Delta v_{m-} - \Delta v_{m+})/I_r] = \omega_m^2 v_m \tag{10}$$

which can be solved economically by Laplace transform. Incidentally, Eq. (10) can also be used for examining the self-adjointness provided the boundary terms $(\Delta v'_{m-} - \Delta v'_{m+})$ and $(\Delta v_{m-} - \Delta v_{m+})$ are incorporated in the operator $\tilde{L}$ in Eq. (7) through the Dirac delta function and its derivatives. This will not be pursued here; the technique, however, can be learned from Ref. 15, pp. 153-156.

The approach displayed above for modal analysis is not generally used; however, it is eminently suitable for our purposes. Before we undertake to manufacture the gyroscopic modes, it is prudent to analyze a spinning but nongyroscopic spacecraft. This is done next.

III. Spinning Unconstrained Modes

Planar dynamics of the spinning structures can be analyzed in essentially the same way as the nonspinning structures. (Here we ignore the important issue of steady-state deformation due to spinning; several papers have been devoted to this end. See, for example, Likins et al.[15,16]) For an illustration, treat the model of the previous section as a gravity-stabilized satellite that revolves around Earth in a circular orbit with an angular veloity Ω (see Fig. 2). In the stable equilibrium position $0x_l$ the appendage $\mathcal{E}$ is along the local vertical; note that $0x_l y_l z_l$ is an orbital frame. The rigid body $\mathcal{R}$ is arbitrary having moments of inertia A_r, B_r, C_r along the body-fixed x, y, and z axis, respectively; $B_e = C_e = 2\rho\ell^3/3$ and $B = B_r + B_e$, $C = C_r + C_e$ are the principal moments of inertia of $\mathcal{E}$ and $\mathcal{V}$ about the y and z axis, respectively. The moment of inertia A is defined similarly.

Our interest lies in the attitude $\theta(t)$ of $\mathcal{R}$ and in the transverse beam deformations $v(x,t)$, both in the orbital plane with positive directions as shown in Fig. 2. Only antisymmetric deformations will be considered here; consequently, we only need to account for the range $+0 \leq x < \ell$ and $\mathcal{R}$. After some basic relations from Meirovitch,[17] the following equations of motion and boundary conditions can be derived

$$C\ddot{\theta} + 3\Omega^2(B-A)\theta + 2\int_{+0}^{\ell} x(\ddot{v} + 3\Omega^2 v)\,\mathrm{d}m = 0 \tag{11}$$

$$+0<x<\ell\colon\; k\Delta k\Delta v - \frac{\partial}{\partial x}[3\Omega^2(\ell^2 - x^2)v'/2] + \ddot{v} + x(\ddot{\theta} + 3\Omega^2\theta) = 0 \tag{12a}$$

$$x = +0\colon\; v = v' = 0; \qquad x = \ell\colon\; \Delta v = \Delta v' = 0 \tag{12b}$$

The absence of the gyroscopic terms in these equations is evident.

The displacement of any point $P\epsilon\mathcal{V}$ with respect to the uniformly spinning axes $0x_l y_l z_l$ is given by

$$w(x,t) = 0, P\epsilon\mathcal{R}; \qquad w(x,t) \triangleq x\theta(t) + v(x,t) \tag{13}$$

To express the dynamics of $\mathcal{R}$ and $\mathcal{E}$ in the frame $0x_l y_l z_l$, Eqs. (11) and (12) can be recast as

$$C_r\ddot{\theta} + 3\Omega^2(B_r - A_r)\theta + 2\int_{+0}^{\ell} x(\ddot{w} + 3\Omega^2 w)\,\mathrm{d}m = 0 \tag{14a}$$

$$+0<x<\ell\colon\; \ddot{w} - 3\Omega^2[(\ell^2 - x^2)w']'/2 + k\Delta k\Delta w = 0 \tag{14b}$$

$$x = \ell\colon\; \Delta w = \Delta w' = 0; \qquad x = +0\colon\; w = 0, w' = \theta \tag{14c}$$

Note the nonhomogeneous boundary condition in Eq. (14c)

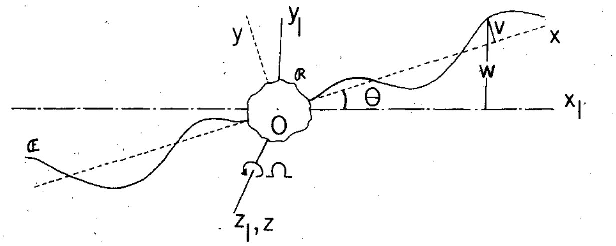

Fig. 2 Spinning, nongyroscopic deformed spacecraft.

where θ is governed by Eq. (14a); this nonhomogeneity turns out to be a key for a proper modal analysis.

Commenting on the existence of a rotational rigid mode in the present circumstances, we note that the vehicle is endowed with a restoring mechanism due to the gravity-gradient torques. Since a rigid mode necessarily has zero frequency so as not to create any acceleration field to act on the elastic structure and thereby not induce any deformation, the vehicle represented by Eq. (14) will *not* have any rotational rigid mode relative to the frame $0x_l y_l z_l$. This is in contradistinction to the situation in Sec. II where the model had a rotational rigid mode. Incidentally, there is an obvious mistake in Ref. 20 where a nonzero frequency is associated with a rigid mode. For a mathematical analysis to prove the nonexistence of a rigid mode in the case of a spinning satellite in a zero-gravity environment the reader is directed to Ref. 9. Thus, presently, the following expansion for a modal analysis is complete

$$w \triangleq \sum_{m=1}^{\infty} w_m(x)\eta_m(t), \qquad w_m(x) = x\theta_m + v_m(x) \tag{15}$$

Simultaneous expansion of v and θ in Eq. (15) as in Sec. II is implied. Substitution of Eq. (15) in Eq. (14) yields the following eigenvalue problem:

$$+0<x<\ell\colon\; k\Delta k\Delta w_m - 3\Omega^2[(\ell^2 - x^2)w'_m]'/2 = \omega_m^2 w_m \tag{16}$$

$$x = 0\colon\; w_{m+} = 0, w'_{m+} = \theta_m; \quad x = \ell\colon\; \Delta w_m = \Delta w'_m = 0 \tag{17}$$

The θ_m in boundary conditions (17) is governed by

$$2(\omega_m^2 - 3\Omega^2)\int_{+0}^{\ell} x w_m(x)\,\mathrm{d}m = \theta_m[3\Omega^2(B_r - A_r) - \omega_m^2 C_r] \tag{18}$$

which is obtained from Eq. (14a).

For the expansion Eq. (15) to be meaningful, we have to prove orthogonality of the mode m to the mode n where $m \neq n$. For this, Eq. (18) requires reshaping. Placing Eq. (16) into Eq. (18), integrating by parts, and performing a little algebra yield

$$-2EI\Delta w_{m+} = [\omega_m^2 C_r - 3\Omega^2(B_r - A_r)]\theta_m \tag{19}$$

which embodies a balance between the bending moment exerted on $\mathcal{R}$ by the deformation of $\mathcal{E}$ and the "inertial" torque exhibited by $\mathcal{R}$. The presence of the eigenvalue ω_m^2 in Eq. (19) should be noted. To appreciate the effect of the spin on the boundary condition, Eq. (19) may be compared with Eq. (6). Anticipating the results of the next section we state that the gyroscopicity will result in profound changes in Eq (19).

To prove the orthogonality, we define

$$L \triangleq k\Delta k\Delta - \frac{3\Omega^2}{2}\frac{\partial}{\partial x}\left[(\ell^2 - x^2)\frac{\partial}{\partial x}\right] \tag{20}$$

from Eq. (16) so that by combining Eq. (19) with Eq. (16) we get

$$\tilde{L}w_m = \omega_m^2 \tilde{w}_m \tag{21}$$

where

$$\tilde{w}_m \triangleq \begin{bmatrix} w_m \\ C_r^{1/2}\theta_m \end{bmatrix},$$

$$\tilde{L}w_m \triangleq \begin{bmatrix} Lw_m \\ \hline C_r^{-1/2}[\theta_m \cdot 3\Omega^2(B_r - A_r) - 2EI\Delta w_{m+}] \end{bmatrix} \tag{22}$$

Define the inner product

$$\langle \tilde{w}_n, \tilde{L}w_m \rangle \triangleq \int_{\mathcal{E}} w_n L w_m \mathrm{d}m$$

$$+\theta_n[\theta_m \cdot 3\Omega^2(B_r - A_r) - 2EI\Delta w_{m+}] \tag{23}$$

With the preliminaries, Eqs. (19-23), we can prove that

$$\langle \tilde{w}_n, \tilde{L}w_m \rangle = \langle \tilde{L}w_n, \tilde{w}_m \rangle \tag{24}$$

which proves the self-adjointness of the mode m and produces the orthonormality conditions

$$\langle \tilde{w}_n, \tilde{w}_m \rangle = \int_{\mathcal{E}} w_m w_n \mathrm{d}m + C_r \theta_m \theta_n = \delta_{mn} \tag{25a}$$

$$\langle \tilde{w}_n, \tilde{L}w_m \rangle = \omega_m^2 \delta_{mn} \tag{25b}$$

Now that the orthogonality of the modes has been established, the following time-dependence of the modes is valid

$$\ddot{\eta}_m + \omega_m^2 \eta_m = 0 \tag{26}$$

3quation (26), of course, has already been used while nanipulating Eqs. (15-25). Rouch and Kao[21] and Hodges et ıl.[22] are helpful in solving Eq. (21).

The reader is reminded that we have performed the modal ınalysis of the whole vehicle at once. We next address ›tructures experiencing the gyroscopic forces.

IV. Gyroscopic Unconstrained Modes

The objective of this section is to characterize the vehicle nodes of a flexible gyroscopic spacecraft. We know[4] that an :xpansion such as Eq. (15) in a configuration space does not 'urnish the orthogonal gyroscopic modes. Consequently, we vill attack the problem with an involved strategy. For con-reteness in the analysis, we choose the previous model and let t spin about the longitudinal axis z by Ω. (Note the change of he symbol from x to z; see Fig. 3.) The analysis proceeds as ollows.

Formualtion of Motion Equations

In this section we assume that the rigid body is inertially ymmetric about the z axis such that A, A, C are the moments ›f inertia of $\mathcal{V}$ about the x, y, and z axis, respectively; the axis ›xyz is a body-fixed frame. The attitude of $\mathcal{R}$ is expressed in ɜrms of the Euler angles θ_x, θ_y. Any point $(0,0,z)$ on the ›eam $\mathcal{E}$ in the frame $Ox_Iy_Iz_I$ before deformation is found at $u,v,z)$ in the frame $Oxyz$ after deformation. The frame $x_Iy_Iz_I$ is a uniformly spinning triad representing the quilibrium state of the vehicle.

To model the dynamics, define the following complex ariables

$$(t) \triangleq \theta_x(t) + i\theta_y(t), \quad q(z,t) \triangleq u(z,t) + iv(z,t), \quad i = \sqrt{-1} \tag{27}$$

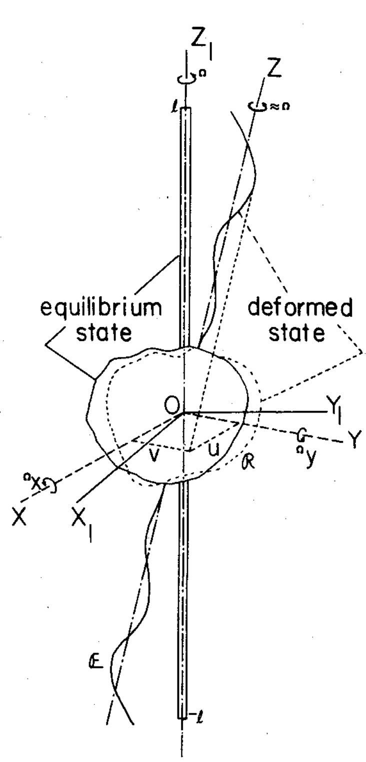

Fig. 3 Gyroscopic spacecraft in vibration.

With Eq. (27) the motion equations of the spin-stabilized satellite deforming antisymmetrically are (see Meirovitch and Nelson[23])

$$A[\ddot{\theta} + i\Omega(1-K)\dot{\theta} + \Omega^2 K\theta] + 2i\int_{+0}^{\ell} z(\ddot{q} + 2i\Omega\dot{q} - \Omega^2 q)\mathrm{d}m = 0 \tag{28}$$

$$+0 < z < \ell: \quad k\Delta k\Delta q + (\ddot{q} + 2i\Omega\dot{q} - \Omega^2 q)$$

$$-iz(\ddot{\theta} + 2i\Omega\dot{\theta} - \Omega^2\theta) = 0 \tag{29a}$$

$$z = +0: \quad q = q' = 0, \qquad z = \ell: \quad \Delta q = \Delta q' = 0 \tag{29b}$$

where $K = (C-A)/A$.

From the standpoint of modal analysis Eqs. (28) and (29) require a series of refinements. To begin, we note that $q(z,t)$ is the deformation of $\mathcal{E}$ in the principal frame $Oxyz$ and is analogous to $v(x,t)$ in Sec. II and III. In the triad $Ox_Iy_Iz_I$, the same point in $\mathcal{E}$ will be

$$Q(z,t) \triangleq q(z,t) - iz\theta(t) \tag{30}$$

where $Q(z,t)$ is analogous to $w(x,t)$ in the previous sections. Repeatedly applying the property of inertial symmetry of $\mathcal{R}$ in the spin plane, Eq. (30) reduces Eqs. (28) and (29) to

$$A_r(\ddot{\theta} + 2i\Omega\dot{\theta} - \Omega^2\theta) + i\int_{\mathcal{E}} z(\ddot{Q} + 2i\Omega\dot{Q}$$

$$-\Omega^2 Q)\mathrm{d}m - iC_r\Omega(\dot{\theta} + i\Omega\theta) = 0 \tag{31}$$

$$+0 < z < \ell: \quad k\Delta k\Delta Q + \ddot{Q} + 2i\Omega\dot{Q} - \Omega^2 Q = 0 \tag{32a}$$

$$z = 0: \quad Q = 0, Q' = -i\theta; \qquad z = \ell: \quad \Delta Q = \Delta Q' = 0 \tag{32b}$$

where the nonhomogeneous boundary condition in Eq. (32b) should be noted. The gyroscopic terms in Eqs. (28, 29, 31, and 32) are evident.

To further simplify Eqs. (31) and (32a), we migrate from the rotating triad $0x_Iy_Iz_I$ to an inertial triad by the following transformations

$$Q_I(z,t) \triangleq Q(z,t)\exp i\Omega t \triangleq q_I(z,t) - iz\theta_I(t) \tag{33}$$

where the subscript I indicates an inertial frame. Using Eq. (33), Eqs. (31) and (32) reduce to

$$A_r\ddot{\theta}_I + i\int_{\mathcal{E}} z\ddot{Q}_I\mathrm{d}m - iC_r\Omega\dot{\theta}I = 0 \tag{34}$$

$$+0<z<\ell: \quad k\Delta k\Delta Q_I + \ddot{Q}_I = 0 \tag{35a}$$

$$z=+0: \quad Q_I=0, \quad Q_I' = -i\theta_I; \quad z=\ell: \quad \Delta Q_I = \Delta Q_I' = 0 \tag{35b}$$

where the last term in Eq. (34) is gyroscopic. The motion equations are now in an elegant form.

Since no torques are acting on the satellite, the angular momentum vector is invariant. Evidently, from Eq. (34) the angular momentum H is

$$H(t) \triangleq A_r\dot{\theta}_I + i\int_{\mathcal{E}} z\dot{Q}_I\mathrm{d}m - iC_r\Omega\theta_I \tag{36}$$

The facility with which an integral of motion H can be deduced from Eq. (34), as compared to deducing one from Eqs. (28) and (31), is striking. Equation (36) will be required subsequently.

Modal Analysis in Phase Space

Likins et al.[16] have analyzed a ground-based cantilever beam spinning about its longitudinal axis. Gyroscopic terms appear in this situation, too; yet, because these terms disappear in an inertial frame (a fact not recognized by them), they succeeded in a modal analysis in a configuration space. However, as is evident in Eq. (34), this does not happen in the present case. Consequently, analogous with a discrete analysis,[1-3] we will arrange Eqs. (34) and (35) in a phase space. This requires substitution of Eq. (35a) in Eq. (34), integration by parts, and the observation of Eq. (35b) to arrive at

$$A_r\ddot{\theta}_I - 2iEI\Delta Q_{I+} - iC_r\Omega\dot{\theta}_I = 0 \tag{37}$$

where $\Delta Q_{I+} = \Delta Q(z=+0,t)$.

There is only one useful phase space representation of Eqs. (35) and (37) which will allow establishing the adjointness and orthogonality properties of the *gyroscopic modes*. After some trials we arrive at the following arrangement:

$$\tilde{L}\begin{bmatrix} k\Delta Q_I \\ \dot{Q}_I \end{bmatrix} \triangleq \frac{\partial}{\partial t}\begin{bmatrix} k\Delta Q_I \\ \dot{Q}_I \\ \hline A_r^{1/2}\dot{Q}_{I+}' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & k\Delta \\ -k\Delta & 0 \\ \hline A_r^{-1/2}[2\rho k & iC_r\Omega\partial/\partial z]_+ \end{bmatrix}\begin{bmatrix} k\Delta Q_I \\ \dot{Q}_I \end{bmatrix} \tag{38}$$

The complex operator $\tilde{L}$ includes a differential as well as a boundary operator separated by a partition. The usefulness of this arrangement is affirmed in what follows.

For a phase space modal analysis, we adopt the following expansion

$$\begin{bmatrix} k\Delta Q_I \\ \dot{Q}_I \end{bmatrix} = \sum_m \begin{bmatrix} G_{mU} \\ G_{mL} \end{bmatrix}\alpha_m(t), \quad \sum_m = \sum_{m=1}^{\infty} \tag{39}$$

The question of whether the expansion in Eq. (39) is complete arises. Note that we are not considering the translational motion of $\mathcal{V}$; therefore, the zero-frequency translational rigid modes are not included in Eq. (39). Also, due to the gyroscopic forces the zero-frequency rotational rigid modes disappear. Thus the expansion in Eq. (39) is complete. Properties of the spatial functions in the right side of Eq. (39) will now be established. It will be proved that these vector functions are indeed the *gyroscopic modes* referred to as the third family of modes in the first section.

The expansion, Eq. (39), transforms Eq. (38) to

$$\sum_m \tilde{G}_m(z)\dot{\alpha}_m(t) = \sum_m \tilde{L}\tilde{G}_m\alpha_m(t) \tag{40}$$

where

$$\tilde{G}_m \triangleq \begin{bmatrix} G_m(z) \\ \hline A_r^{1/2}G_{mL+}' \end{bmatrix}, \quad G_m(z) \triangleq \begin{bmatrix} G_{mU}(z) \\ G_{mL}(z) \end{bmatrix} \tag{41}$$

If we take the time behavior of $\alpha_m(t)$ such that

$$\dot{\alpha}_m(t) = i\omega_m\alpha_m(t) \tag{42}$$

the eigenvalue problem deduced from Eq. (40) is

$$\tilde{L}G_m = \omega_m\tilde{G}_m \tag{43}$$

To render $\tilde{G}_m$ unique, its boundary conditions are established by recalling Eqs. (35b, 37, 39, 41, and 42) which leads to

$$z=+0: \quad G_{mL+}=0, \quad z=\ell: \quad G_{mU}=G_{mU}'=0 \tag{44a}$$

$$z=+0: \quad i(\omega_mA_r - \Omega C_r)G_{mL+}' = 2\rho kG_{mU+} \tag{44b}$$

Though these boundary conditions are useful in their present form, their true nature is unveiled if we observe that

$$k\Delta G_{mL}(z) = i\omega_mG_{mU}(z) \tag{45}$$

which changes Eq. (44b) to

$$[(i\omega_m)^2A_r - (i\omega_m)(i\Omega C_r)]G_{mL+}' = 2EI\Delta G_{mL+} \tag{46}$$

The reader may now compare Eq. (46) with Eqs. (18) and (6). The distinctive feature of Eq. (46) is that due to gyroscopicity ($i\Omega C_r$) the eigenvalue $i\omega_m$ appears linearly as well as quadratically, whereas in the previous cases the eigenvalue ω_n^2 appeared with at best a spin-induced constant. The structure of Eq. (46) precludes the formation of the gyroscopic mode in a configuration space. Also, the time behavior, Eq. (42) should be compared with that in Eq. (26).

Regarding the adjointness properties of the operator $\tilde{L}$ and the vector function $\tilde{G}_m$ we require a complex-type scalar product in a Hilbert space (see Chaps. 1, 4 in Ref. 15). Let there be an adjoint mode $\tilde{F}_n$ which has entries similar to Eqs (41). By the definition of the above inner product we have

$$\langle\tilde{F}_n,\tilde{L}G_m\rangle \triangleq \int_{\mathcal{E}}(\bar{F}_{nU}k\Delta G_{mL} - \bar{F}_{nL}k\Delta G_{mU})\mathrm{d}m$$

$$+\bar{F}_{nL+}'(2\rho kG_{mU+} + i\Omega C_rG_{mL+}') \tag{47}$$

where an overbar denotes complex conjugate and

$$\tilde{L}G_m = \left[\begin{array}{c} k\Delta G_{mL} \\ -k\Delta G_{mU} \\ \hline A_r^{-1/2}(2\rho k G_{mU+} + i\Omega C_r G'_{mL+}) \end{array}\right] \tag{48}$$

Integration by parts are performed in Eq. (47). If we take $\tilde{L}F_n$ to be similar to Eq. (48) and if the function F_n obeys the same boundary conditions as G_n, then we can prove that

$$\tilde{F}_n = \tilde{G}_n; \quad \langle \tilde{G}_n, \tilde{L}G_m \rangle = -\langle \tilde{L}G_n, \tilde{G}_m \rangle \tag{49a}$$

Thus we conclude that the adjoint L^* of the operator $\tilde{L}$ is

$$L^* = -\tilde{L} \tag{49b}$$

which implies that the operator $\tilde{L}$ is a skew Hermitian operator; see Ref. 24. Eq. (49) equips us to prove the following, much desired, orthonormality conditions

$$\langle \tilde{G}_n, \tilde{G}_m \rangle = \int_{\mathcal{E}} (\bar{G}_{nu} G_{mu} + \bar{G}_{nL} G_{mL})\,dm + A_r \bar{G}'_{nL+} G'_{mL+} = \delta_{mn} \tag{50}$$

$$\langle \tilde{G}_n, \tilde{L}G_m \rangle = i\omega_m \delta_{mn} \tag{51}$$

The reader is cautioned not to attribute the properties of Eq. (49) to *all* gyroscopic situations. For example, if the vehicle is equipped with an angle-dependent restoring mechanism, the phase space arrangement, Eq. (38), will change dramatically and the property, Eq. (49b), will be replaced by $L^* = -\tilde{L}^T$ where the superscript T means transpose. For details the reader may refer to Hablani.[25]

The next challenging question is concerned with the relation of these modes with the angular momentum H in Eq. (36). We now devote our attention to this aspect.

Angular Momentum and Gyroscopic Modes

The upper part of Eq. (39) combined with Eq. (45) yields

$$\frac{\partial^2}{\partial z^2}\left[Q_I(z,t) - \sum_m \alpha_m(t) G_{mL}(z)/i\omega_m\right] = 0 \tag{52}$$

and the lower part of Eq. (39) with the aid of Eq. (42) furnishes

$$\frac{\partial}{\partial t}\left[Q_I(z,t) - \sum_m \alpha_m(t) G_{mL}(z)/i\omega_m\right] = 0 \tag{53}$$

Equations (52) and (53) along with the boundary condition $Q'_{I+} = -i\theta_I$ lead to

$$Q_I(z,t) = C_H z + \sum_m \alpha_m(t) G_{mL}(z)/i\omega_m \tag{54}$$

$$\theta_I(t) = iC_H + \sum_m \alpha_m(t) G'_{mL+}/\omega_m \tag{55}$$

where C_H is a constant. Also, the lower part of Eq. (39) gives rise to

$$\dot{\theta}_I(t) = i\sum_m G'_{mL+}\alpha_m(t) \tag{56}$$

Turning our attention to the eigenvalue problem Eq. (43) and reviewing Eqs. (41) and (49), we are led to its following simplified version

$$+0 < z < \ell: \quad k\Delta k\Delta G_{mL} = \omega_m^2 G_{mL} \tag{57a}$$

$$z = +0: \quad 2EI\Delta G_{mL+} = (\omega_m \Omega C_r - \omega_m^2 A_r) G'_{mL+} \tag{57b}$$

$$z = +0: \quad G_{mL} = 0, \qquad z = \ell: \quad \Delta G_{mL} = \Delta G'_{mL} = 0 \tag{57c}$$

We are now in a position to formulate a significant result. Substitute Eqs. (56) and (55) and the lower part of Eq. (39) in Eq. (36) to obtain

$$H(0) \triangleq H_0 = C_H \Omega C_r + i\sum_m H_m \alpha_m(t) \tag{58a}$$

where, in the absence of external stimuli, the angular momentum H_0 is conserved and

$$H_m \triangleq \int_{\mathcal{E}} zG_{mL}(z)\,dm + (A_r - C_r\Omega/\omega_m) G'_{mL+} \tag{58b}$$

A placement of Eq. (57a) in (58b), integration by parts, and an observation of the boundary conditions, Eqs. (57b) and (57c), prove that

$$H_m = 0, \qquad m = 1,2,\ldots \tag{59}$$

which states that *the angular momentum associated with the* m*th gyroscopic mode is zero.* Without offering a proof, we claim that the property, Eq. (59), is valid for a gyroscopic mode, as defined here, of *any* gyroscopic spacecraft. One more example of this will be seen in the next section. From Eq. (58a) we thus get

$$C_H = H_0/\Omega C_r \tag{60a}$$

Equation (60a) transforms Eqs. (54) and (55) to

$$Q_I(z,t) = H_0 z/\Omega C_r + \sum_m G_{mL}(z)\alpha_m(t)/i\omega_m \tag{61a}$$

$$\theta_I(t) = iH_0/\Omega C_r + \sum_m G'_{mL+}\alpha_m(t)/\omega_m \tag{61b}$$

Thus, in Eqs. (58a) and (61) we see a decomposition of dynamics into two parts: 1) absorption of entire angular momentum by a static inertial slope C_H of the vehicle. This part defines the location of a certain frame identified below and, 2) the zero overall angular momentum carried by the elastic deformation which is expressed as a sum of infinite gyroscopic modes and which is measured from the frame defined by the first part.

For nongyroscopic spacecraft there exists the concept of Tisserand's frame — a frame whose bearings relative to an inertial frame depend on entire momentum of the vehicle, translational as well as angular and relative to which the elastic deformations do not contribute to the overall momentum. In other words, a motion resulting from all the zero-frequency rigid modes defines the Tisserand's frame and the unconstrained elastic modes do not possess any overall linear and angular momentum. For further details the reader may consult Canavin and Likins,[26] Milne,[27] Hughes,[12] and Samin and Willems.[28] An analogous situation exists in Eqs. (58a) and (61). The frame defined by the first terms in the right sides of Eqs. (61) is in fact the Tisserand's frame associated with the gyroscopic spacecraft in hand. In the preceding we have considered a force-free situation. A stimulated situation is considered in the next section where further theory on the dynamics of Tisserand's frame will be enunciated.

In Eq. (60) we have granted a spatial interpretation of C_H. There also exists a time interpretation for C_H; this can be extracted from Eq. (54) by substituting $t=0$. Thus,

$$C_H = -i\theta_{I0} + i\sum_m \alpha_{m0} G'_{mL+}/\omega_m;\quad \theta_{I0} \triangleq \theta_I(0), \alpha_{m0} = \alpha_m(0) \tag{60b}$$

Equation (60b) helps in acquiring a proper understanding of the response. Thus we have accomplished our objective of devising a gyroscopic mode for a flexible spinning spacecraft. The scene is now shifted to a model in which gyroscopicity is due to a rigid symmetric rotor.

V. Dynamics of a Spacecraft with Stored Angular Momentum

We now analyze the previous model so modified that the rigid body houses a rotor with an angular momentum h spinning along the longitudinal axis $0z$. The vehicle itself does not spin; therefore, the subscript I is henceforth dropped. Hughes and Sharpe[29] have studied the influence of the angular momentum on the frequency; we will study the influence on the modes. In this section the rigid body is taken to be a sphere of radius a; see Fig. 4.

To show the additional value of the gyroscopic modes we will consider torquers (T_{rx}, T_{ry}) on $\mathcal{R}$ and symmetrically located pair (T_{ex}, T_{ey}) at $(0,0,z')$ and $(0,0,-z')$ on $\mathcal{E}$. Based on the definitions Eqs. (27), (30), and

$$T_r \triangleq T_{rx} + iT_{ry}, \qquad T_e \triangleq T_{ex} + iT_{ey}, \qquad T \triangleq T_r + 2T_e \tag{62}$$

the following equations of motion and boundary conditions for antisymmetric deformations can be constructed

$$A_r\ddot{\theta} + 2i\int_a^{\ell'} z\ddot{Q}\mathrm{d}m - ih\dot{\theta} = T, \qquad l' \triangleq a+\ell \tag{63}$$

$$a<z<\ell': \quad k\Delta k\Delta Q + \ddot{Q} = i\frac{\partial}{\partial z}\delta(z-z')T_e/\rho, \tag{64}$$

$$z=a: \quad Q=-ia\theta, Q'=-i\theta; \quad z=\ell': \Delta Q = \Delta Q' = 0 \tag{65}$$

The equivalence between the models in Figs. 3 and 4 becomes clear when Eqs. (63-65) are compared with Eqs. (34) and (35). In Eqs. (63-65) the nonzero radius of the rigid hub and the point torquers introduce additional complications: the number of nonhomogeneous boundary conditions increasing to two in Eqs. (65); the derivative of a Dirac delta operator, δ, appearing in Eq. (64). Isolating the dynamics of the rigid body by substituting Eq. (64) in Eq. (63) and by intergrating by parts we get, similar to Eq. (37),

$$A_r\ddot{\theta} + 2iEI(a\Delta Q'_a - \Delta Q_a) - ih\dot{\theta} = T_r \tag{66}$$

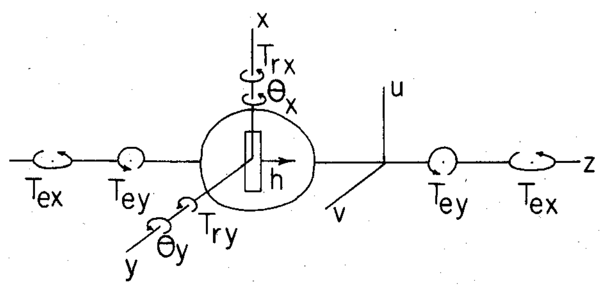

Fig. 4 Spacecraft with stored angular momentum.

where $\Delta Q'_a = \partial^3 Q/\partial z^3\ (z=a;t)$, etc.; the presence of the shear force due to the hub radius a should be noted.

In this section our program is elaborate. Specially, we will derive these two responses of the spacecraft in Fig. 4: 1) when the appendages are treated rigid, and 2) when they are elastic. Next we shall deduce certain modal identities by considering the angular momentum in the presence of external stimuli. These maneuvers will bring the second response closest to the first response.

With regard to the first response, the motion equation which governs the rigid spacecraft can be obtained from Eq. (66) by letting $q(z,t)=0$. This yields

$$A\ddot{\theta}(t) - ih\dot{\theta}(t) = T(t) \tag{67}$$

whose solution, written in a manner which involves the instantaneous angular momentum $H(t)$, is

$$\theta(t) = iH(t)/h + (\theta_0 - iH_0/h)\exp i\omega_p t \\ - ih^{-1}\int_0^t T(\tau)\exp i\omega_p(t-\tau)\mathrm{d}\tau \tag{68}$$

where

$$H(t) = A\dot{\theta}(t) - ih\theta(t) \triangleq H_0 + \int_0^t T(\tau)\mathrm{d}\tau \tag{69}$$

and $\omega_p = h/A$. Note the angular impulse imparted by the stimuli $T(t)$ in Eq. (69). The rigid response Eq. (68) contains a frequency-independent slope $iH(t)/h$ and two terms depending on the frequency ω_p. Now, let us appreciate that when the appendages are treated elastic the rigid mode having frequency ω_p in Eq. (68) will disappear for the reasons discussed before Eq. (15). Then, how the response Eq. (68) is modified due to the elasticity of $\mathcal{E}$ is the subject of the remainder of this section. A question of tremendous interest is whether the term $iH(t)/h$ survives in elastic response.

Phase Space Modal Analysis

Equations (64) and (66) can be arranged in a phase space form such that

$$\frac{\partial}{\partial t}\begin{bmatrix} k\Delta Q \\ \dot{Q} \\ \hdashline A_r^{1/2}\dot{Q}'_a \end{bmatrix} = \tilde{L}\begin{bmatrix} k\Delta Q \\ \dot{Q} \end{bmatrix} + \begin{bmatrix} 0 \\ i\dfrac{\partial}{\partial z}\delta(z-z')T_e/\rho \\ \hdashline -iA_r^{-1/2}T_r \end{bmatrix} \tag{70}$$

where

$$\tilde{L} \triangleq \begin{bmatrix} 0 & k\Delta \\ -k\Delta & 0 \\ \hdashline -2A_r^{-1/2}\rho k\left(a\dfrac{\partial}{dz}-1\right)_a & iA_r^{-1/2}h\left(\dfrac{\partial}{\partial z}\right)_a \end{bmatrix} \tag{71}$$

To construct the gyroscopic modes the expansion Eq. (39) is substituted in the homogeneous counterpart of Eq. (70). This leads to the eigenvalue problem of Eq. (43) where

$$\tilde{G}_m \triangleq \begin{bmatrix} G_m(z) \\ \hline A_r^{1/2} G'_{mLa} \end{bmatrix} \tag{72}$$

and $\tilde{L}$ and $G_m(z)$ are defined in Eqs. (71) and (41), respectively. Due to finite hub radius boundary conditions obeyed by G_{mU} and G_{mL} have a complicated appearance; specifically, we have

$$a<z<\ell': \quad k\Delta k\Delta G_{mL} = \omega_m^2 G_{mL}$$

$$z=a: \quad G_{mLa} = aG'_{mLa}, \quad 2EI(a\Delta G'_{mLa} - \Delta G_{mLa})$$

$$= (-\omega_m h + \omega_m^2 A_r) G'_{mLa}$$

$$z=\ell': \quad \Delta G_{mL} = \Delta G'_{mL} = 0 \tag{73}$$

which may be compared with Eq. (57). Note the linearity of $G_{mL}(z)$ in the range $0<z<a$ in the condition of Eqs. (73b). With these relations available the reader can prove the adjoint property of Eq. (49b) and the orthonormality conditions of Eqs. (50) and (51); the fact that we are using a complex-type inner product should be recalled.

Attention is now drawn to the simulated dynamics represented by Eq. (70). Shortly, we will require the following property of the Dirac delta function

$$\int_a^{\ell'} \phi(z) \frac{\partial}{\partial z} \delta(z-z')\,dz = -\phi'(z') \tag{74}$$

The expansion Eq. (39), the orthonormality conditions Eqs. (50) and (51) and the property Eq. (74) transform Eq. (70) to the following equation for the modal coordinate $\alpha_m(t)$

$$\dot{\alpha}_m = i\omega_m \alpha_m - i[\bar{G}'_{mLa} T_r(t) + 2\bar{G}'_{mLA} T_e(t)] \tag{75}$$

where $\bar{G}'_{mLA} \triangleq \bar{G}mL(z')$; the subscript A connotes actuator. Equation (75) has the solution

$$\alpha_m(t) = \alpha_{m0} \exp i\omega_m t - i\int_0^t [2\bar{G}'_{mLA} T_e(\tau)$$

$$+ \bar{G}'_{mLa} T_r(\tau)] \exp i\omega_m (t-\tau)\,d\tau \tag{76}$$

As a side observation we note that Eq. (75) is the counterpart of the second-order modal coordinate time equation deduced by Hughes and Skelton[30] to define the modal controllability of a *simple* spacecraft; the right side of Eq. (75) quantifies the controllability $\mathcal{C}_m$ (a 2×1 row matrix) of the phase space mode $\tilde{G}_m$, that is,

$$\mathcal{C}_m \triangleq [\bar{G}'_{mL}(a) \quad 2\bar{G}'_{mL}(z')] \tag{77}$$

Thus a physical interpretation is now available for the modal controllability of a gyroscopic spacecraft.

Returning to the question of the response $Q(z,t)$, with the aid of Eq. (75) the lower part of Eq. (39) yields

$$\frac{\partial}{\partial t}[Q(z,t) + i\sum_m G_{mL}(z)\alpha_m(t)/\omega_m]$$

$$= \sum_m G_{mL}(z)[2\bar{G}'_{mLA} T_e(t) + \bar{G}'_{mLa} T_r(t)]/\omega_m \tag{78}$$

Integration of Eq. (78) leads to

$$Q(z,t) = f(z) + \sum_m G_{mL}(z)[H_r(t)\bar{G}'_{mLa}$$

$$+ H_e(t)\bar{G}'_{mLA} - i\alpha_m(t)]/\omega_m \tag{79}$$

where $H_r(t)$ and $H_e(t)$ are the angular impulses added by the torquers

$$H_r(t) \triangleq \int_0^t T_r(\tau)\,d\tau \qquad H_e(t) \triangleq \int_0^t 2T_e(\tau)\,d\tau \tag{80}$$

The unknown function $f(z)$ is determined just as we established the linear term in the right side of Eq. (61a); thus,

$$f(z) = H_0 z/h \tag{81}$$

In the process we discover the following zero angular momentum property of the mode $\boldsymbol{G}_m$

$$H_m = 2\int_a^{\ell'} zG_{mL}\,dm + (A_r - h/\omega_m)G'_{mLa} = 0 \tag{82}$$

which is analogous to the H_m in Eq. (58b). The time-interpretation of $f(z)$ can be deduced by applying Eq. (79) for the instant $t=0$; thus,

$$f(z) = Q(z,0) + i\sum_m G_{mL}(z)\alpha_{m0}/\omega_m \tag{83}$$

It is amazing to learn how a simple linear function, Eq. (81), can have an exactly equivalent representation, Eq. (83), which involves an infinite series.

Our chief interest, however, is the attitude of the rigid body; this can be derived from Eq. (79) by recalling that $\theta(t) = iQ'(a,t)$. Thus

$$\theta(t) = iH_0/h + i\sum_m G'_{mLa}[H_r(t)\bar{G}'_{mLa}$$

$$+ H_e(t)\bar{G}'_{mLA} - i\alpha_m(t)]/\omega_m \tag{84}$$

where Eq. (81) has been utilized. When we compare the elastic response, Eq. (84), with the rigid response, Eq. (68), we notice vast differences. Some of these differences may be eliminated if we probe the workings of the angular momentum; this we do next.

Evolution of Angular Momentum

From motion equation (63) we note that the angular momentum $H(t)$ is

$$H(t) = A_r\dot{\theta}(t) + 2i\int_a^{\ell'} z\dot{Q}(z,t)\,dm - ih\theta(t) \tag{85}$$

Our immediate goal is to express the right side of Eq. (85) in terms of the modes $\boldsymbol{G}_m(z)$ and the modal coordinates $\alpha_m(t)$, $m=1,2,\ldots$. To accomplish this we modify Eq. (56) to become

$$\dot{\theta}(t) = i\sum_m G'_{mLa}\alpha_m(t) \tag{86}$$

We set Eq. (86), the bottom portion of Eq. (39) and the response, Eq. (84) in Eq. (85) and capitalize on the zero angular momentum property Eq. (82) of the mode $\tilde{G}_m(z)$; this operation transforms the right side of Eq. (85) to

$$H(t) = H_0 + hH_r(t)\sum_m \bar{G}'_{mLa}G'_{mLa}/\omega_m$$

$$+ H_e(t)h\sum_m \bar{G}'_{mLA}G'_{mLa}/\omega_m \tag{87}$$

On the other hand, we know that

$$H(t) = H_0 + H_r(t) + H_e(t) \tag{88}$$

By comparing Eq. (87) with Eq. (88) we discover two identities—one for $T_r(t)$ and the other for $2T_e(t)$:

$$h \sum_m \bar{G}'_{mLa} G'_{mLa}/\omega_M = 1, \qquad h \sum_m \bar{G}'_{mLa} G'_{mLa}/\omega_m = 1 \tag{89}$$

These results are astonishing in that the coefficients of disturbability, G_{mLa} and G_{mLA}, in the two left sides are different and yet the sums of the two infinite series are the same. These identities have been computationally verified in Ref. 31.†

The identities in Eq. (89) simplify the response of Eq. (84) to

$$\theta(t) = iH(t)/h + \sum_m G'_{mLa}\alpha_m(t)/\omega_m \tag{90}$$

Inserting $\alpha_m(t)$ from Eq. (76) in the response of Eq. (90), it adopts the following form:

$$\theta(t) = iH(t)/h + \sum_m \alpha_{m0} G'_{mLa} e^{i\omega_m t}/\omega_m$$
$$- i \sum_m G'_{mLa} \bar{G}'_{mLA} \int_0^t 2T_e(\tau) e^{i\omega_m(t-\tau)} \mathrm{d}\tau/\omega_m$$
$$- i \sum_m G'_{mLa} \bar{G}'_{mLa} \int_0^t T_r(\tau) e^{i\omega_m(t-\tau)} \mathrm{d}\tau/\omega_m \tag{91}$$

The response, Eq. (91) has the closest resemblance possible with the rigid response, Eq. (68). Commenting on the Tisserand's frame, the first term, $iH(t)/h$, in both Eqs. (68) and (91) defines the instantaneous inclination of the frame in an inertial system. Thus, our research culminates in this abstraction: *The dynamics of the Tisserand's frame is the same whether the gyroscopic spacecraft is rigid or elastic.* We have detected the same pattern also in the case of a wholly elastic two-dimensional rectangular structure with rotors.[25] Consequently, we speculate that the above abstraction must be valid for all gyroscopic spacecraft. Further, it is a genralization of an identical observation in the nongyroscopic situations.

To complete the present efforts, the initial value $\alpha_m(0)$ in Eq. (91) will now be linked to the actual initial conditions. Note that we are working with the modal expansion

$$\begin{bmatrix} k\Delta Q(z,t) \\ \dot{Q}(z,t) \\ \hdashline A_r^{1/2}\dot{Q}'(a,t) \end{bmatrix} = \sum_m \tilde{G}_m(z)\alpha_m(t) \tag{92}$$

An inner product of Eq. (92) with $\tilde{G}_n$ $m \neq n$, and the use of the orthogonality property, Eq. (50), lead to, for $t = 0$.

$$\alpha_{m0} = 2\int_a^{\ell'} [\bar{G}_{mU}(z) k\Delta Q(z,0)$$
$$+ \bar{G}_{mL}(z)\dot{Q}(z,0)]\mathrm{d}m - iA_r\bar{G}'_{mLa}\dot{\theta}(0) \tag{93}$$

†Subsequent research has shown that the function $G_{mL}(z)$, $m = 1 \ldots, \infty$, is indeed real; see Ref. 31 for details.

The absence of $\theta(0)$ in Eq. (93) is because the vehicle does not have any angle-dependent restoring torque [see Eqs. (63-65)].

VI. Concluding Remarks

The dominant contribution in this work is an exact theory of the vehicle modes for gyroscopic spacecraft, entitled "gyroscopic modes." Though concrete examples of space vehicles have been attacked, the techniques shown are applicable to continuum models of any spacecraft. We summarize our findings thus:

1) Characterization of vehicle modes even for nongyroscopic spacecraft requires a careful mathematical analysis; the complications are due to the central rigid body which causes boundary conditions to involve eigenvalues.

2) The vehicle modes for spinning, nongyroscopic spacecraft are obtained with as much difficulty, or ease, as for the nonspinning, nongyroscopic spacecraft; however, in the former case the question of existence of zero frequency rigid modes warrants close attention.

3) A special care is called for to discover a proper phase space arrangement of a continuum model of a gyroscopic spacecraft in order to be able to determine the adjoint and the orthogonality properties of the gyroscopic modes. This feature is in contrast with the relative ease in manipulating discrete gyroscopic systems.

4) Just as in a nongyroscopic spacecraft, dynamics of Tisserand's frame for a gyroscopic spacecraft is the same, whether the vehicle is treated as rigid or as elastic; this conclusion remains valid even in the presence of stimuli. To arrive at this abstraction, however, requires detailed analysis.

5) In a gyroscopic situation the conventional zero frequency rotational rigid modes do not exist; the spacecraft instead continuously seeks an inclination in an inertial frame of an amount which accounts for its instantaneous angular momentum. Elastic deformation or the gyroscopic modes relative to this inclined frame known as Tisserand's frame do not contribute to the overall angular momentum of the vehicle.

We believe that the analytical tools offered here will prove to be useful to the analysts encountering a gyroscopic spacecraft.

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